Vibrational Motion Chapter 19

- Why should soldiers not march in step when they go over a bridge?
- Why do you need to "pump" your legs when you begin swinging on a park swing?
- How can you carry a full cup of coffee without splashing it?

Be sure you know how to:

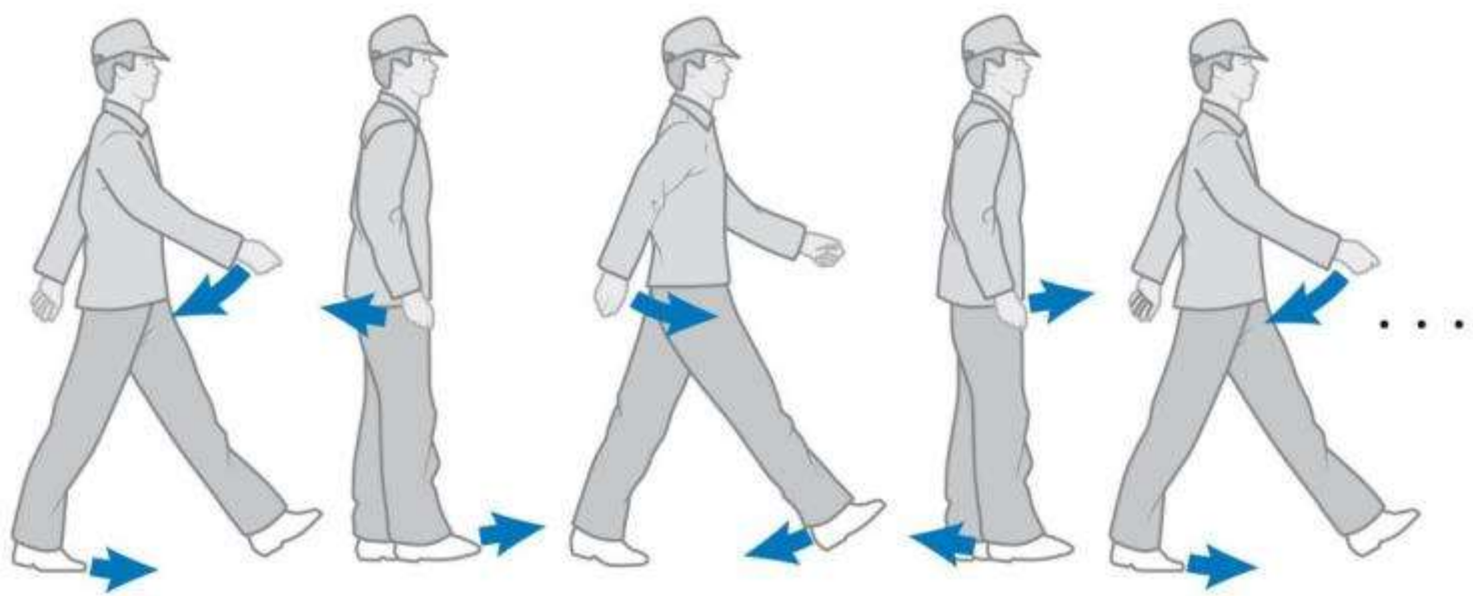
- Apply Hooke's law to analyze forces exerted by springs (Section 6.4).
- Use radians to describe angles (Section 8.1).
- Draw an energy bar chart for a process and convert it into an equation (Sections 6.2 and 6.6).

What's new in this chapter

- We have studied linear motion—objects moving in straight lines at either constant velocity or constant acceleration.
- We have also studied objects moving at constant speed in a circle.
- In this chapter we encounter a new type of motion, in which both direction and speed change.

Observations of vibrational motion

- When you walk, your arms and legs swing back and forth. These motions repeat themselves.

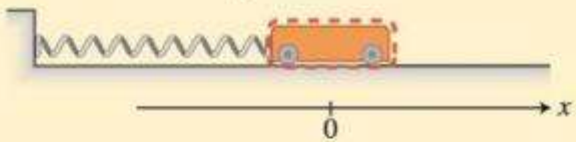
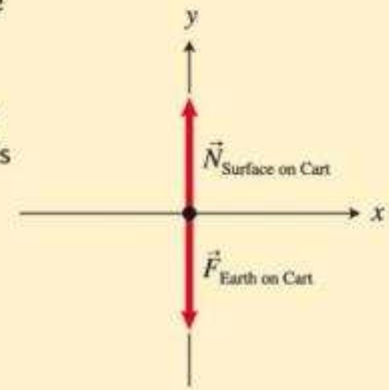
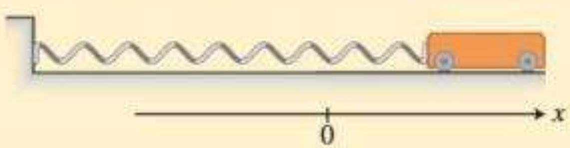
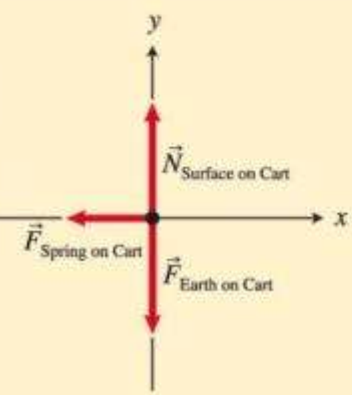


- The back-and-forth motion of an object that passes through the same positions is an important feature of vibrational motion.

Observational experiment

OBSERVATIONAL EXPERIMENT TABLE

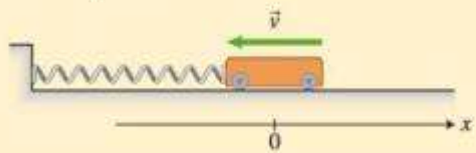
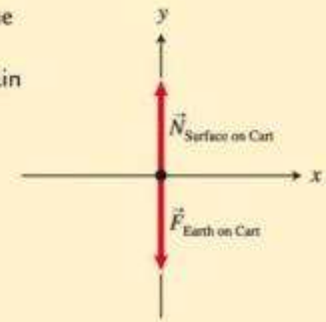
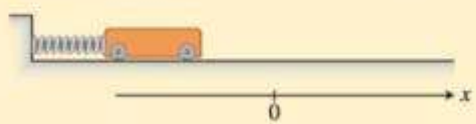
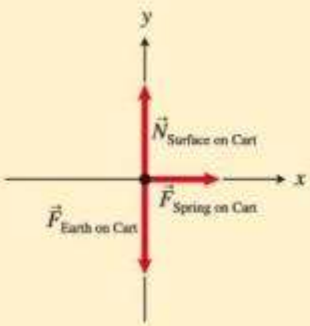
19.1 Some features of vibrational motion.

Observational experiment	Analysis
Experiment 1. A cart attached to a relaxed spring sits at rest on a horizontal surface at position 0. 	In all experiments, we choose the cart as the system. The spring is relaxed and the sum of the forces exerted on the cart by other objects is zero when at position 0. 
Experiment 2. (a) Now pull the cart to the right and release it. The cart starts to move back toward position 0. 	The spring exerts a force on the cart toward the left. The sum of the forces exerted on the cart now points to the left and causes the cart to start moving left toward position 0. 

Observational experiment

OBSERVATIONAL EXPERIMENT TABLE

19.1 Some features of vibrational motion. *(Continued)*

Observational experiment	Analysis
(b) The cart is moving fast when it reaches position 0 and overshoots that position. 	As the cart passes position 0, the sum of the forces that other objects exert on the cart is again zero. But since it is moving, it continues moving. 
(c) The cart now slows down and eventually stops to the left of position 0, then starts moving back to the right toward position 0. 	After passing position 0, the spring exerts a force to the right toward position 0; the sum of the forces points to the right, causing the cart to slow down, stop, and move back toward position 0. 
(d) The cart overshoots again and eventually stops where it started, on the right side of 0. The motion then repeats itself.	

Patterns

The system is at rest in a particular position when not vibrating. When the cart is at rest in this position, the sum of the forces that other objects exert on it is zero. When the cart is displaced from this position and released, the cart moves back and forth, passing through that position in two opposite directions. If displaced from the rest position, a force is exerted on the vibrating object that tends to return it to that position.

Equilibrium position

Equilibrium position (or just **equilibrium**) The position at which a vibrating object resides when not disturbed. When resting at this position, the sum of the forces that other objects exert on it is zero. During vibrational motion the object passes back and forth through this position from two opposite directions.

Restoring force

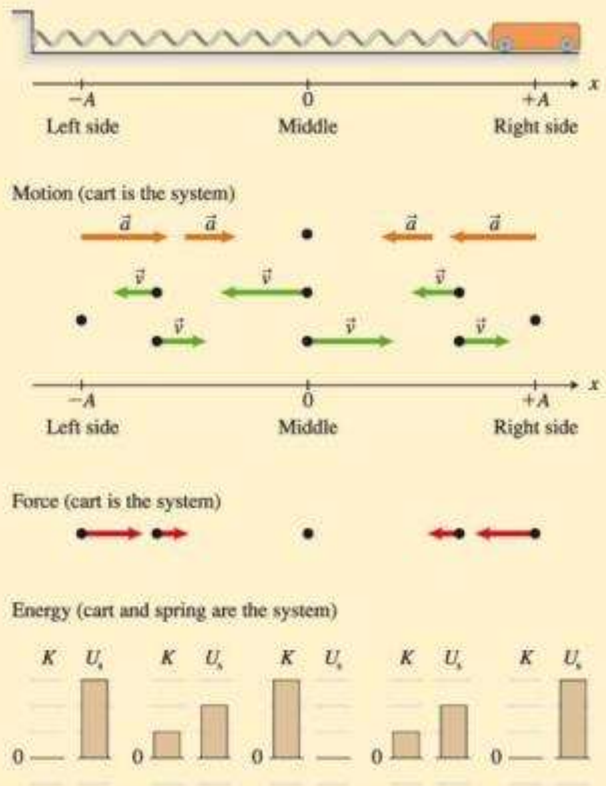
Restoring force When an object is displaced from equilibrium, some other object exerts a force with a component that always points opposite the direction of the vibrating object's displacement from equilibrium. This force tends to restore the vibrating object back toward equilibrium.

Observational experiment

OBSERVATIONAL EXPERIMENT TABLE

19.2 **Multiple representation analysis of a cart on a spring.**

VIDEO 19.2

Observational experiment	Analysis
A cart attached to a spring is pulled to the right (position +A) and released. It moves past equilibrium (position 0) at high speed and briefly stops an equal distance to the left of equilibrium (position −A). The spring pulls the cart back toward the right. The process repeats over and over. We analyze the process using motion diagrams, force diagrams, and energy bar charts.	

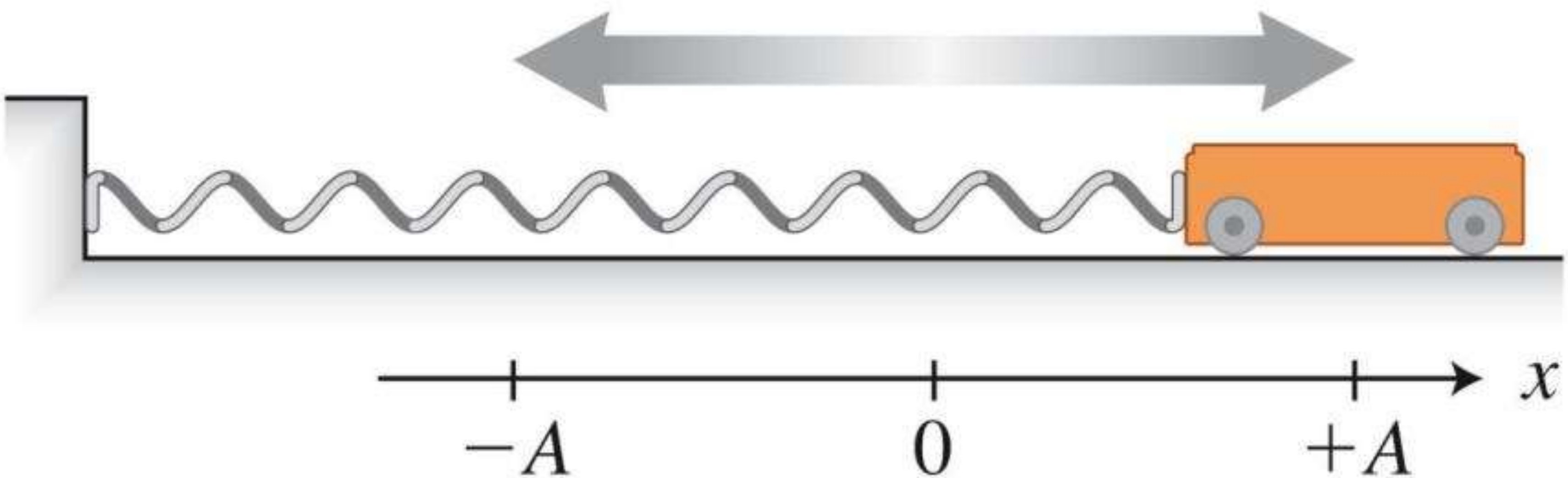
Pattern

Restoring force The restoring force is zero as the vibrating object passes through the equilibrium position and has maximum magnitude when at the extreme positions on the left and right.

Potential and kinetic energy The energy of the vibrating system (the cart and spring) varies between maximum potential energy when at the extreme positions to maximum kinetic energy as the object passes equilibrium. In between, the energy is a combination of kinetic and potential energy.

Amplitude

Amplitude The amplitude A of a vibration is the magnitude of the maximum displacement of the vibrating object from its equilibrium position.



Patterns observed in vibrational motion

- An object passes through the same positions, moving first in one direction and then in the opposite direction.
- The object passes the equilibrium position at high speed. When it overshoots, a restoring force exerted on it by some other object points back toward equilibrium.
- A system composed of the vibrating object and the object exerting the restoring force has maximum potential energy when at extreme positions and maximum kinetic energy at equilibrium.

Period

Period The period T of a vibrating object is the time interval needed for the object to make one complete vibration—from the clock reading when it passes through a position while moving in a certain direction until the next clock reading when it passes through that *same* position moving in the *same* direction. The unit of period is the second.

Frequency

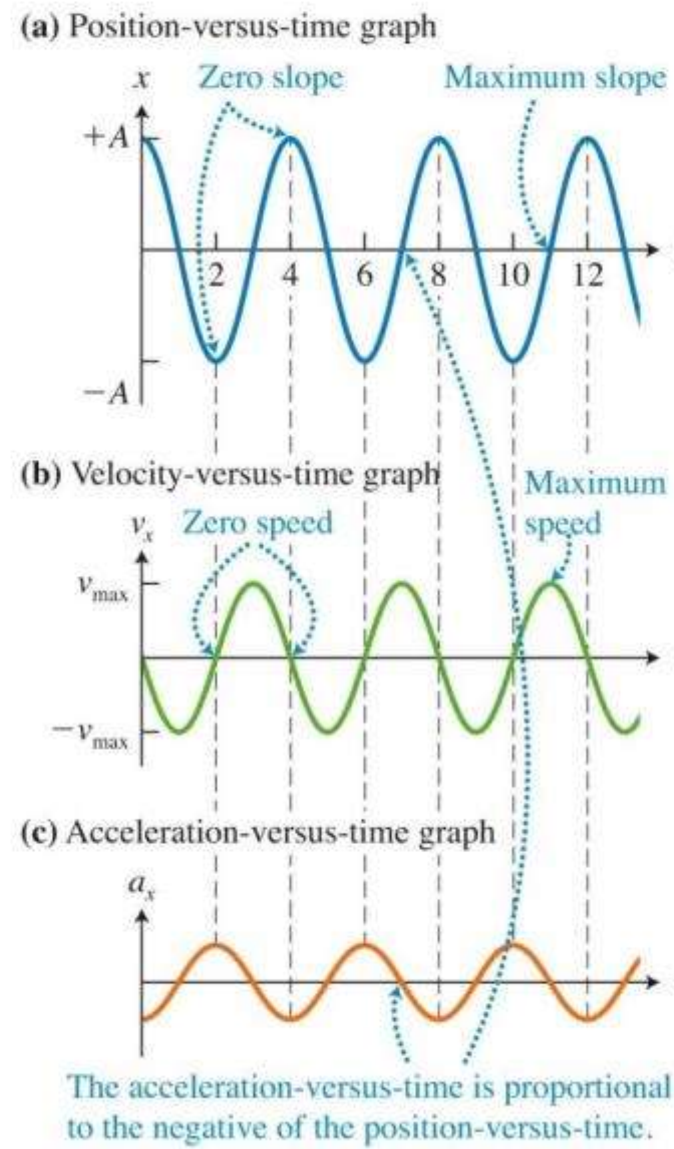
Frequency The frequency f of vibrational motion is the number of complete vibrations of the system during one second. Frequency is related to period:

$$f = \frac{1}{T} \quad (19.1)$$

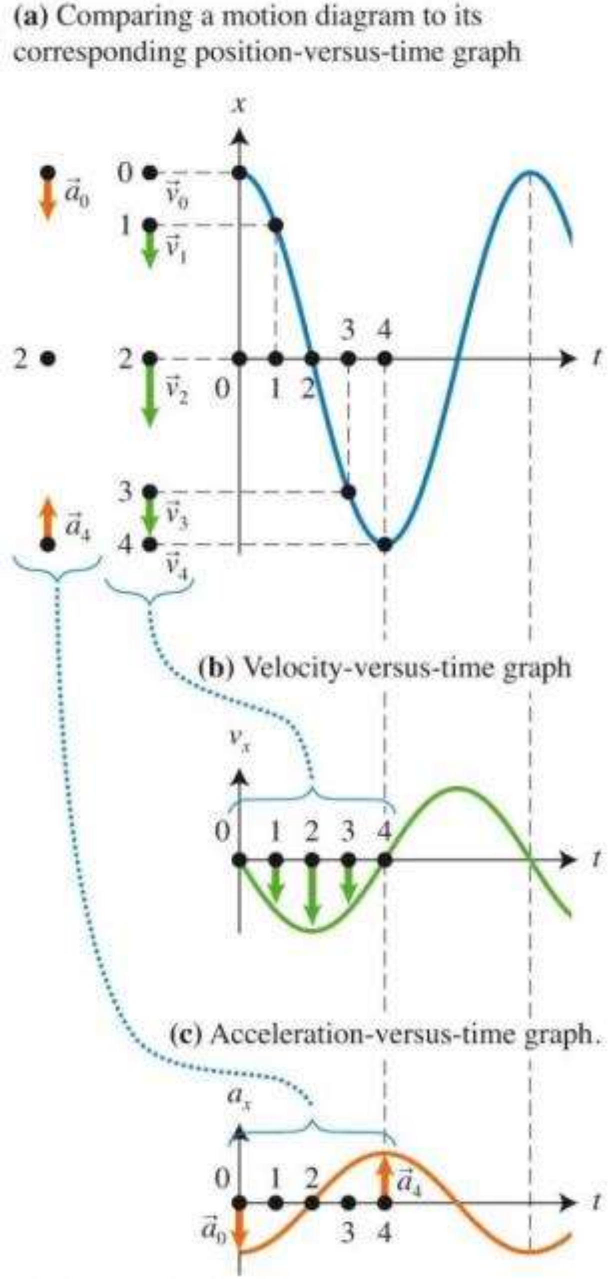
The unit for frequency is the hertz (Hz), where $1\ \text{Hz} = 1\ \text{vib/s} = 1\ \text{s}^{-1}$.

Kinematics of vibrational motion

- In an experiment, a motion detector collects position, velocity, and acceleration-versus-time data for a cart vibrating on a spring.

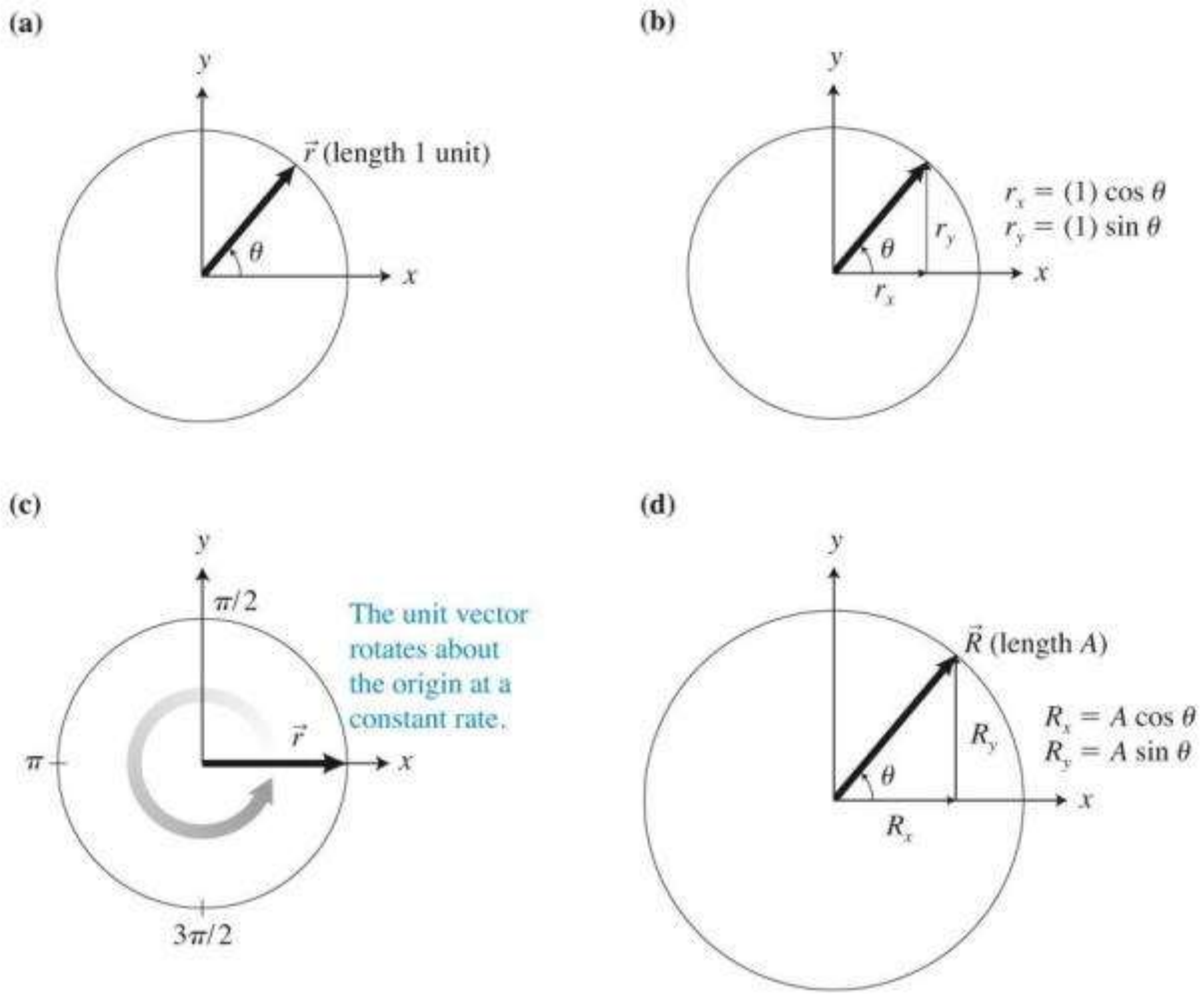


Consistency of motion diagram and graphs



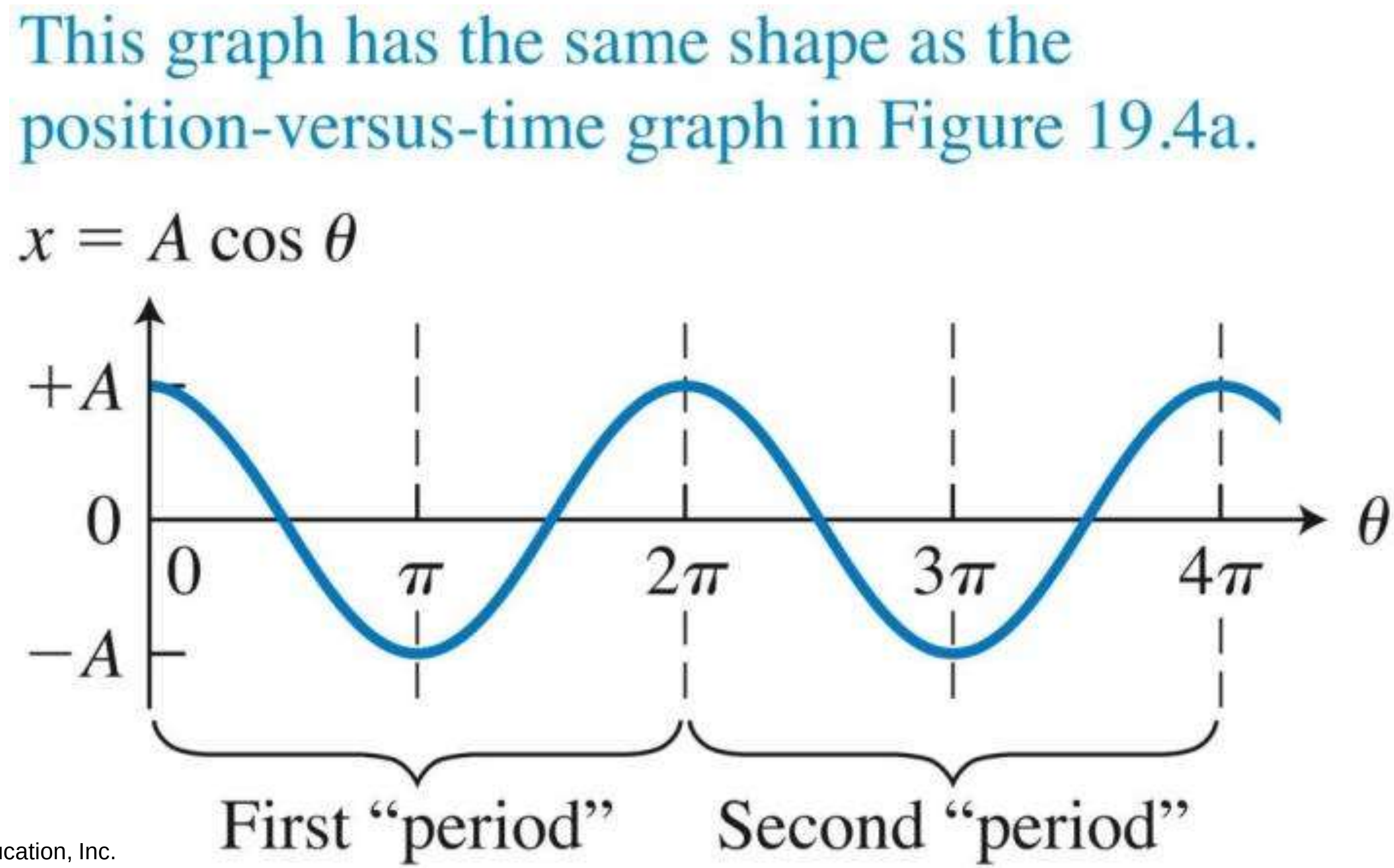
Mathematical description of position as a function of time

- The meanings of sine and cosine can be best understood using a unit circle.



Mathematical description of position as a function of time

- A graph of $x = A \cos(\theta)$ looks very similar to the position-versus-time graph produced by the motion detector for a cart on a spring.



Mathematical description of position as a function of time

- We can write the period function $x(t)$ to represent the position-versus-time graph:

$$x = A\cos\left(\frac{2\pi}{T}t\right)$$

- Notice that $x = +A$ at $t = 0$. If an object is at $x = 0$ at $t = 0$, you can either adjust the cos function by adding $-(\pi/2)$ or use the sine function.

Simple harmonic motion

- Simple harmonic motion (SHM) is motion that can be described by the following equation:

$$x = A\cos\left(\frac{2\pi}{T}t\right)$$

- It is a mathematical model of motion.

Position of a vibrating object as a function of time

Table 19.3 Position of a vibrating object as a function of time.

Clock reading t of the vibrating object shown in Figure 19.4a	Position x of the vibrating object shown in Figure 19.4a	Angle of the radius vector θ (radians) for the function $x = A\cos\theta$	Value of the function $x = A\cos\theta$
0 (0 s)	A	0	A
$T/4$ (1 s)	0	$\pi/2$	0
$T/2$ (2 s)	$-A$	π	$-A$
$3T/4$ (3 s)	0	$3\pi/2$	0
T (4 s)	A	2π	A
$2T$ (8 s)	A	4π	A
$3T$ (12 s)	A	6π	A

Mathematical description of velocity and acceleration as a function of time

- If the position function is given by:

$$x = A\cos\left(\frac{2\pi}{T}t\right)$$

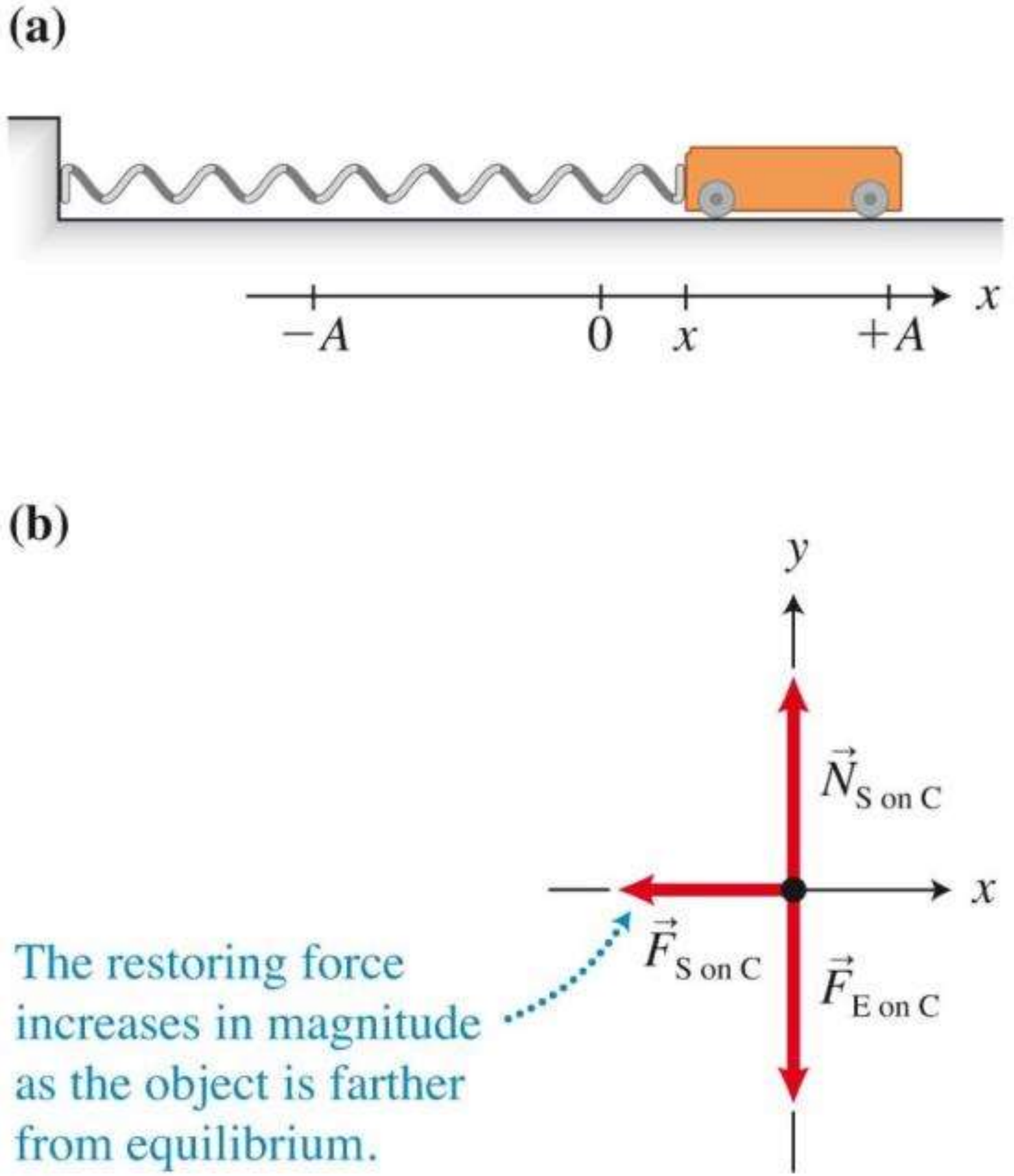
- Then the velocity and acceleration functions are:

$$v_x = -\left(\frac{2\pi}{T}\right)A\sin\left(\frac{2\pi}{T}t\right)$$

$$a_x = -\left(\frac{2\pi}{T}\right)^2 A\cos\left(\frac{2\pi}{T}t\right)$$

- *A* is the amplitude of the vibration; *T* is the period of the vibration.

The dynamics of simple harmonic motion



Forces and acceleration for a cart on a spring

- According to Hooke's law, the force that the stretched spring exerts on a cart in the *x*-direction is:

$$F_{\text{S on C}\,x} = -kx$$

- Using Newton's second law, we get:

$$a_x = \frac{-kx}{m} = -\frac{k}{m}x$$

- The cart's acceleration a_x is proportional to the negative of its displacement x from the equilibrium position.

Period of vibrations of a cart attached to a spring

- Starting with:

$$a_x = \frac{-kx}{m} = -\frac{k}{m}x$$

- And using:

$$x = A\cos\left(\frac{2\pi}{T}t\right) \qquad a_x = -\left(\frac{2\pi}{T}\right)^2 A\cos\left(\frac{2\pi}{T}t\right)$$

- We get:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

- In this expression for period, there is no dependency on the amplitude.

The frequency of vibration of an object attached to a spring

- Using the equation $T = 2\pi\sqrt{\frac{m}{k}}$ and the relationship $f = 1/T$, we find:

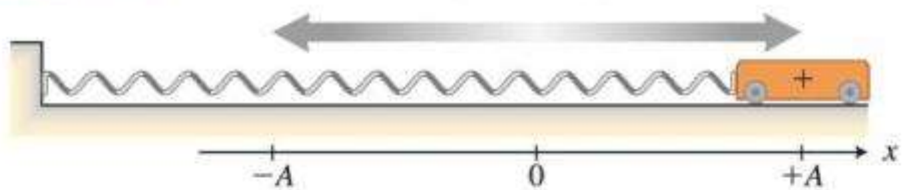
$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

- In our derivation, we assumed that the spring obeys Hooke's law, that the spring has zero mass, and that the cart is a point-like object.
- We also neglected friction.

Energy of vibrational systems

- As a cart-spring system vibrates, the energy of the system continuously changes from all elastic to all kinetic.

Table 19.4 Variation of energy during one vibration.



Clock reading t	Displacement	Elastic potential energy U_s	Kinetic energy K	Total energy U_{tot}
$\frac{1}{2}T$	$-A$	$\frac{1}{2}kA^2$	0	$U_{tot} = \frac{1}{2}kA^2$
$\frac{1}{4}T$	0	0	$\frac{1}{2}mv_{max}{}^2$	$U_{tot} = \frac{1}{2}mv_{max}{}^2$
$\frac{3}{4}T$	0	0	$\frac{1}{2}mv_{max}{}^2$	
0	A	$\frac{1}{2}kA^2$	0	$U_{tot} = \frac{1}{2}kA^2$
T	A	$\frac{1}{2}kA^2$	0	

Relationship between the amplitude of the vibration and the cart's maximum speed

- The equation $U = \frac{1}{2}kA^2 = \frac{1}{2}mv_{max}^2 = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$ can be rearranged to give:

$$v_{max} = \sqrt{\frac{k}{m}}A$$

- This makes sense conceptually:
  - When the mass of the cart is large, it should move slowly.
  - If the spring is stiff, the cart will move more rapidly.

Tip

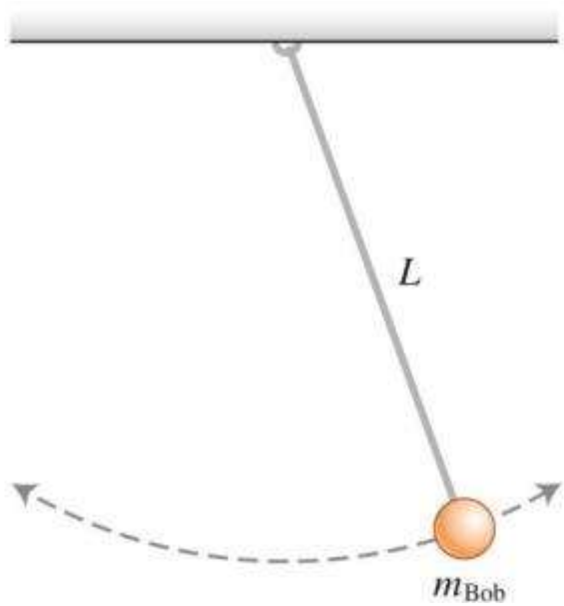
TIP In the above discussion we neglected the interactions of the system with the surface of the track and with the air. These would both do negative work on the system and gradually decrease its energy, eventually bringing the vibrating system to rest.

Example 19.5

- A spring with a 1.6×10^4 N/m spring constant and a 0.1-kg cart at its end has a total vibrational energy of 3.2 J.
  1. Determine the amplitude of the vibration.
  2. Determine the cart's maximum speed.
  3. Determine the cart's speed when it is displaced 0.010 m from equilibrium.
  4. What would the amplitude of the vibration be if the energy of the system doubled?

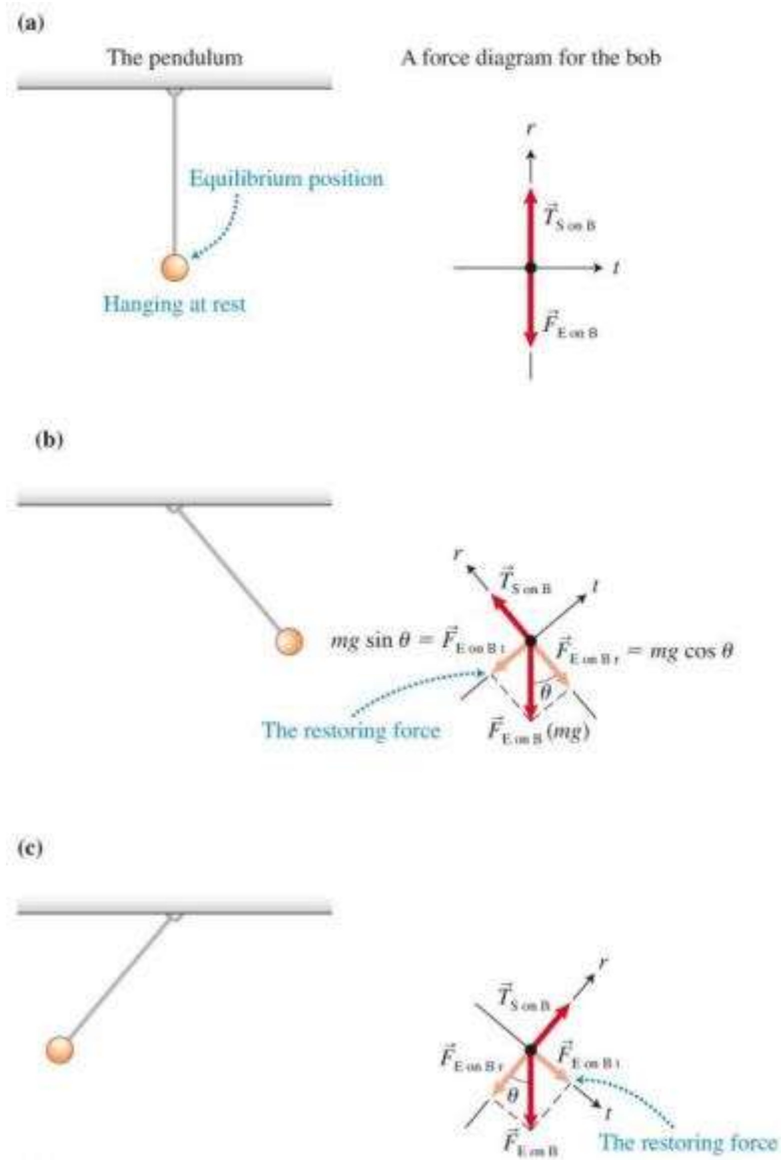
The simple pendulum

- A pendulum is a vibrating system in which the motion is very apparent.
- Consider a simplified model of a pendulum system that has a compact object (a bob) at the end of a comparatively long and massless string and that undergoes small-amplitude vibrations.
  - This idealized system is called a simple pendulum.



The simple pendulum

- Two objects interact with the bob of the pendulum.
  - The string S exerts a force that is always perpendicular to the path of the bob.
  - Earth exerts a downward gravitational force.

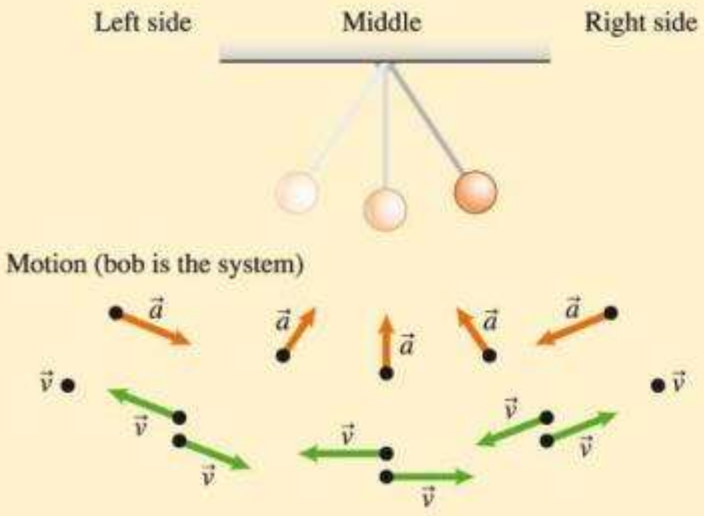


Observational experiment

OBSERVATIONAL EXPERIMENT TABLE

19.5 Multiple representation analysis of the pendulum.

VIDEO 19.5

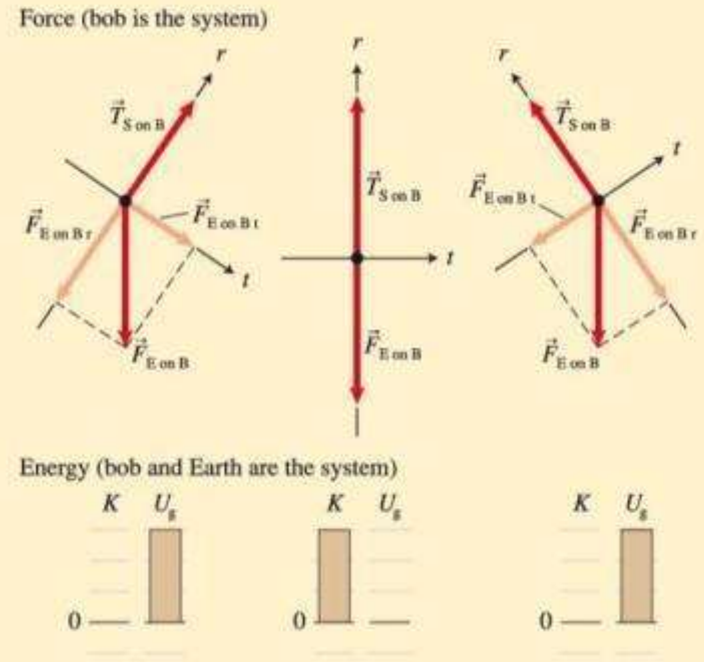
Observational experiment	Analysis
A bob hanging from a string is pulled to the right and released. It swings down past the equilibrium position at high speed and stops on the other side the same distance to the left that it started on the right. The tangential component of the gravitational force pulls the bob back toward equilibrium. It overshoots and then returns to its starting position. The process repeats over and over.	

Observational experiment

OBSERVATIONAL EXPERIMENT TABLE

19.5 **Multiple representation analysis of the pendulum.** ***(Continued)***

VIDEO 19.5

Observational experiment	Analysis
	

Pattern

Restoring force The restoring force is zero as the vibrating object passes through the equilibrium position and has maximum magnitude when at the extreme positions on the left and right sides.

Potential and kinetic energy The energy of the vibrating system varies between maximum potential energy when at the extreme positions to maximum kinetic energy as the object passes equilibrium. In between, the energy is a combination of kinetic and potential energy.

The simple pendulum in relation to the cart on a spring

- The motion of the pendulum has the same patterns as the motion of the cart on a spring:
  - It passes the equilibrium position from two different directions.
  - There is a restoring force exerted on the bob.
  - The system's energy oscillates between maximum potential and maximum kinetic.

Experimental investigation of the period of a simple pendulum

- We record a simple pendulum's period T for different vibration amplitudes A, bob masses m, and string lengths L.

Table 19.6 Effect of bob mass, string length, and amplitude on pendulum period.

Bob mass m (kg)	String length L (m)	Amplitude θ (°)	Period T (s)
1.0	1.0	10.0	2.0
2.0	1.0	10.0	2.0
3.0	1.0	10.0	2.0
1.0	1.0	10.0	2.0
1.0	**2.0**	10.0	2.8
1.0	**3.0**	10.0	3.5
1.0	**4.0**	10.0	4.0
1.0	1.0	**15.0**	2.0
1.0	1.0	**20.0**	2.0
1.0	1.0	**25.0**	2.0

- The period appears to depend only on the string length.

Deriving the period of a simple pendulum

- The tangential component of the restoring force is:

$$F_{\text{E on Bt}} = -mg\,\theta = -mg\left(\frac{x}{L}\right) = -\left(\frac{mg}{L}\right)x$$

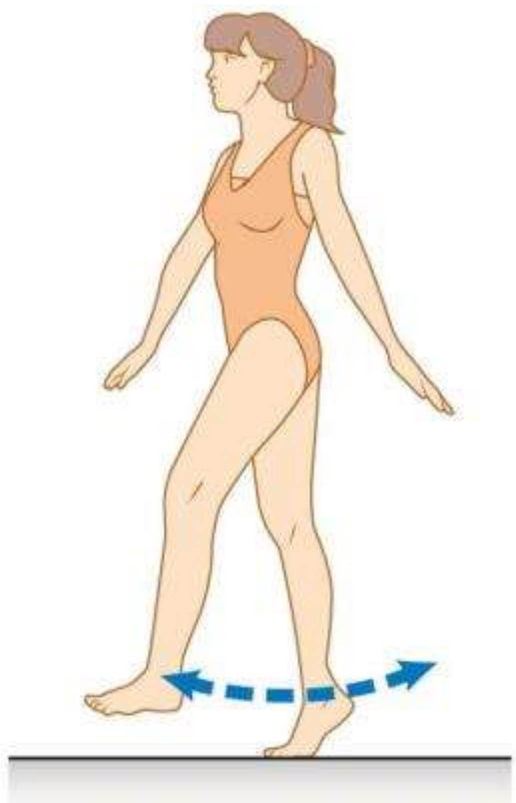
- The restoring force is directly proportional to the bob's displacement from equilibrium, so the period of a pendulum is:

$$T = 2\pi\sqrt{\frac{L}{g}}$$

- The period does not depend on the mass or the amplitude.

Example 19.6

- Estimate the number of steps that a leg takes each second while a person is walking—in other words, the natural swinging frequency of the leg. We will treat the leg as a simple pendulum.



Conceptual Exercise 19.7

- A child sits on a swing that hangs straight down and is at rest. Draw energy bar charts for the child-swing-Earth system:
    1. As a person pulls the child back in preparation for the first swing.
    2. At the moment the person releases the swing while it is at its elevated position.
    3. As the swing passes the equilibrium position.
    4. When the swing reaches half the maximum height on the other side.
    5. As the swing passes the equilibrium position moving in the opposite direction

Skills for analyzing processes involving vibrational motion

- When problem solving:
  - Represent the process with force diagrams and/or bar charts if needed.
  - If necessary:
    - Use kinematics equations to describe the changing motion of the object.
    - Use force diagrams to apply the component form of Newton's second law to the problem or use bar charts to apply work-energy principles.
    - Use the expressions for the period of an object attached to a spring or to a pendulum.

Example 19.9

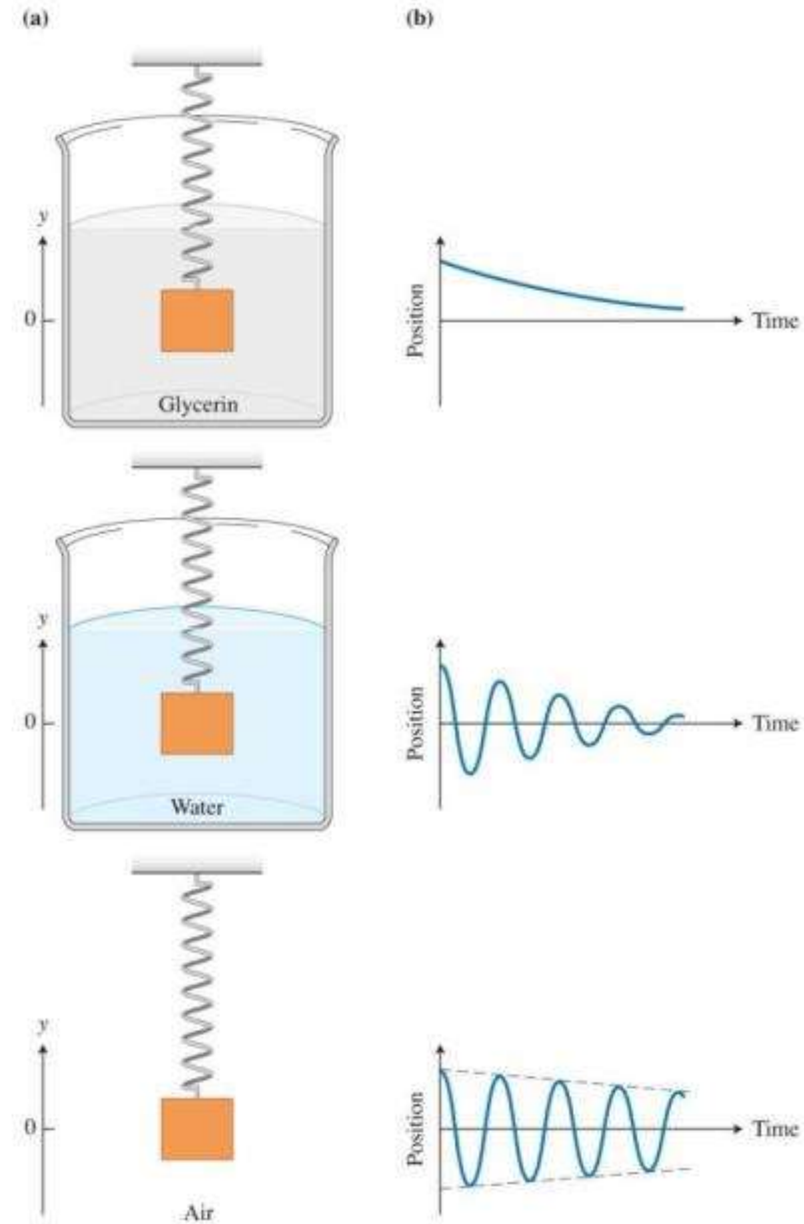
- The Body Mass Measurement Device chair (mass = 32 kg) has a vibrational period of 1.2 s when empty. When an astronaut sits on the chair, the period changes to 2.1 s. Determine:
  - The effective spring constant of the chair's spring.
  - The mass of the astronaut.
  - The maximum vibrational speed of the astronaut if the amplitude of vibration is 0.10 m.

Including friction in vibrational motion

- So far we have mostly ignored the effect of friction on vibrating objects.
  - Without friction, a car would continue vibrating on its suspension system for miles after going over a bump on a road, and tall buildings would continue to sway even after the wind had died down.

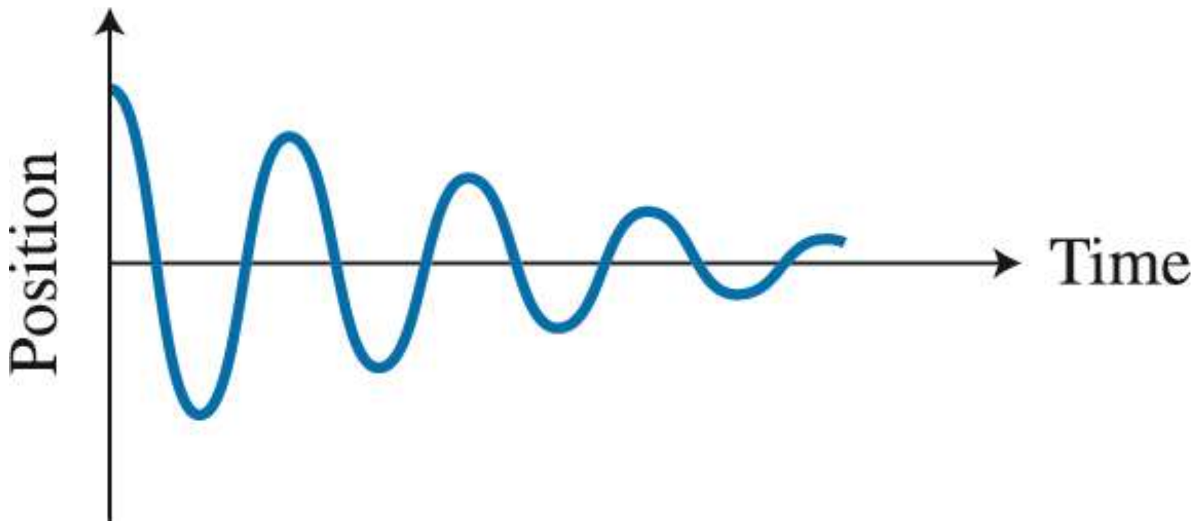
Damped and undamped oscillators

- You can observe the effects of friction on a simple system.



Damping

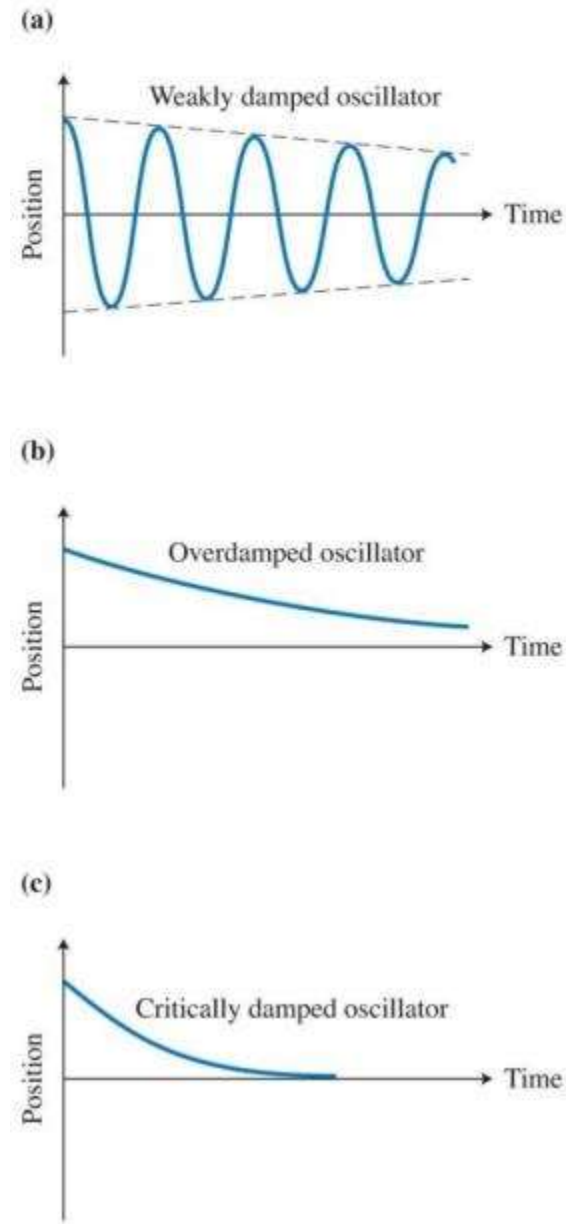
- The phenomenon of decreasing vibration amplitude and increasing period is called damping.



- Damping is a useful aspect of the design of vehicles and bridges.

Including friction in vibrational motion

- A weakly damped system continues to vibrate for many periods.
- In an overdamped system, the vibrating system takes a long time to return to the equilibrium position, if it ever does.
- In a critically damped system, the vibrating object returns to equilibrium in the shortest time possible.



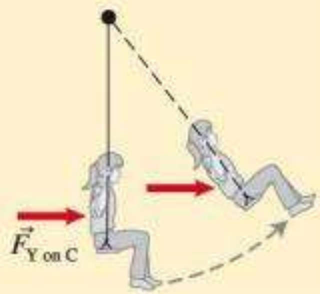
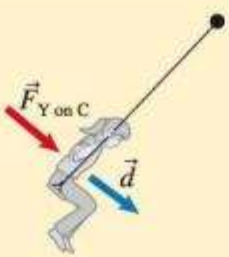
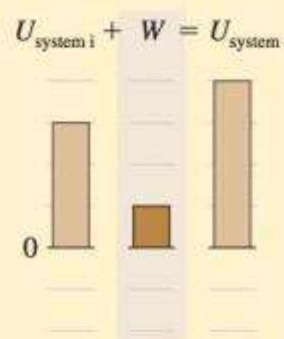
Vibrational motion with an external driving force

- All real vibrations are damped and eventually stop unless energy is added to the system.
- We will investigate how external interactions of the environment with a vibrating system could lead to continuous vibrations.

Observational experiment

OBSERVATIONAL EXPERIMENT TABLE

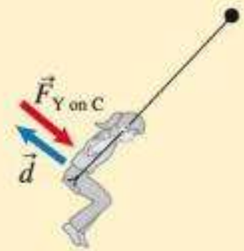
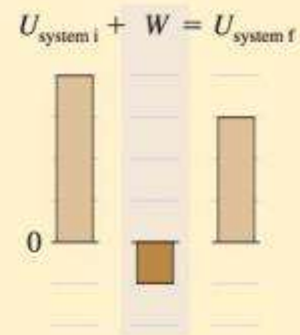
19.7 A forced vibration.

Observational experiment	Analysis
Experiment 1. A young child sits on a swing. You exert a steady force on the child and swing until the swing cable is at such an angle that the swing stops (this angle depends on the construction of the swing). There are no vibrations. 	The swing, child, and Earth are the system (but not you). You exerted a constant force that did positive work on the swing. But no vibrations occurred during this time.
Experiment 2. A young child swings with a period $T = 2\pi\sqrt{\frac{L}{g}}$. You push her gently and briefly every time just after she reaches the peak of her vibration. Each brief push is in the same direction as her displacement. The pushes have the same period as her swinging. 	You exerted a variable force that did positive work on the swing. The energy of the system increased. The amplitude of the swing vibrations increases. 

Observational experiment

OBSERVATIONAL EXPERIMENT TABLE

19.7 A forced vibration. *(Continued)*

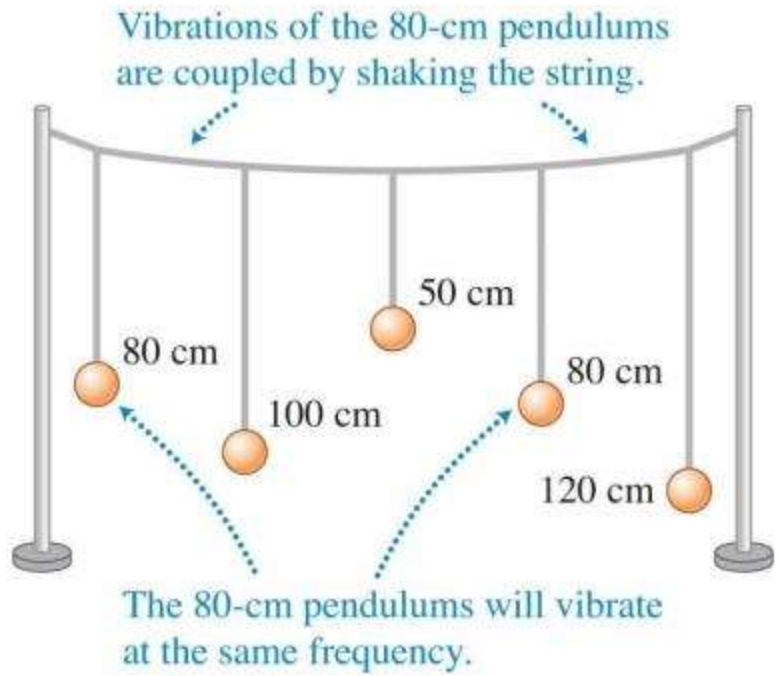
Observational experiment	Analysis
Experiment 3. The child still swings with period $T = 2\pi\sqrt{\frac{L}{g}}$. This time you push her gently and briefly as she swings back toward you, just before she reaches the peak of her vibration. You push opposite the direction of her displacement. 	You did negative work on the swing. The energy of the system decreased. The amplitude of the swing vibrations decreases. 
Experiment 4. The child still swings with period $T = 2\pi\sqrt{\frac{L}{g}}$. This time you push her gently with a different period. Sometimes you push as she is moving back and at other times as she is moving forward. Her amplitude becomes small and stays small.	You do positive work sometimes and negative work at other times (the net work over time is zero). Friction and air resistance cause the swinging amplitude to decrease.

Pattern

For an external force exerted on the swinging system to cause the amplitude to increase:

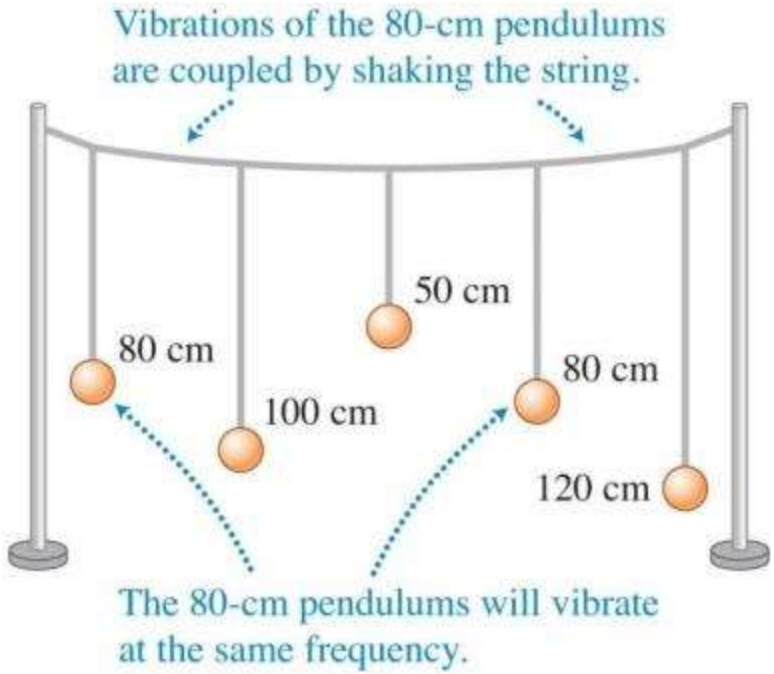
1. The external force has to be a *variable force*.
2. The external force must do *positive work* on the vibrating system.
3. The *period (and therefore frequency)* of the external force *must match* that of the vibrating system.

Resonance



- Resonance occurs when some object in the environment exerts a force on it that varies in time and does net positive work over time.
  - This work increases the total energy of the system and therefore the amplitude of vibrations.
  - The increase in energy occurs when the frequency of the external force is the same as the natural frequency of the system.

Energy transfer through resonance



- Transfer of energy from one object to another depends on two conditions:
  - The natural frequencies of the objects must be close.
  - A mechanism must exist that allows one object to do positive work on the other object.

Energy transfer through resonance

- Resonance caused the collapse of the Tacoma Narrows Bridge in Washington only four months after its completion.

(a)

(b)

Radiation absorption by molecules

- To maintain a consistent climate, Earth and its atmosphere rely on a balanced exchange of energy in and out of the system they represent.
  - The energy absorbed by the Earth-atmosphere system must be balanced by the energy transferred out of that system.
  - The energy is transferred out when vibrating molecules on Earth emit lower-frequency infrared radiation that travels away from Earth.

Greenhouse effect

- Some of the radiation emitted by molecular vibration on Earth is absorbed by CO_2 molecules in the atmosphere.
  - This absorption causes the vibrational energy of these atmospheric molecules to increase.
  - The excited CO_2 molecules re-emit the infrared radiation, much of which is reabsorbed by molecules on Earth.
  - Thus less energy is transferred out of the system than is transferred in—and the planet warms.